# Discontinuous Galerkin for Diffusion

Second Annual and Final Report to AFOSR
regarding Grant FA9550-06-1-0425

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## 1 Foreword

This report regards in detail the research carried out under AFOSR Grant FA9550-06-1-0425, "Discontinuous Galerkin for Diffusion," in the final project period of 9 months (June 1, 2007 - February 29, 2008), and then reviews the entire yield of the project, thus serving as the Final Report..

The original award period for the project ran from 1 June 2006 through 30 November 2007; a 3-month no-cost extension was requested by the PI for medical reasons, and granted.. Thus, in the second year of the project the grant only covered the 9 months from 1 June 2007 through 29 February 2008.

We bring into mind that, as reported in the first annual report covering the period from 1 June 2006 through 31 May 2007, this project on the recovery-based Discontinuous Galerkin (RDG) method for diffusion operators has been rich in early successes. Most of the tasks listed in the 2005 proposal were accomplished in the first year; in addition, an unexpected fundamental concept, the "recovery basis," was developed.

## 2 Accomplishments/new findings in the second year

In the second year the effort addressed the following issues:

1. *Making RDG suitable for multidimensional applications.* This meant coming up with the proper polynomial bases for recovery at the interface between arbitrary triangular or tetrahedral cells. The 2-D rule is described in an AIAA paper by Van Leer, Lo and Van Raalte [1]; the 3-D rule is a nontrivial extension and still awaits publication.

2. *Increasing its efficiency by constructing the recovery basis once as pre-processing step at the start of an evolutionary or steady-state calculation.* At the basis lies the Fundamental Theorem of Recovery by Van Raalte and Van Leer [2], which says that the expansion on two neighboring elements of the discontinuous discrete solution in terms of the original discontinuous basis functions is identical in coefficients to the expansion of the recovered smooth solution in terms of the recovered smooth basis functions (the recovery basis).

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**14. ABSTRACT**
The funded period of this project ran out on 30 November 2007; a no-cost extension of 3 months was requested by the PI for medical reasons, and was granted. In the second year the effort addressed the following issues: 1) making the Recovery-based Discontinuous Galerkin method (RDG) suitable for multidimensional applications, 2) increasing its efficiency by constructing the recovery basis once as pre-processing step for an entire calculation, 3) the combination of RDG with upwind DG for advection (URDG), 4) writing the general 1-D form ofRDG as a penalty method, 5) dissemination via conferences and publications, and 6) technology transfer. Over the total grant period the RDG method developed from a promising one-dimensional Discontinuous Galerkin discretization technique for diffusion terms with superior accuracy and good stability to an efficient multidimensional methodology with a solid theoretical underpinning and ready for transfer to Air Force/industrial use.

**15. SUBJECT TERMS** Diffusion equation, Poisson equation, advection-diffusion equation, Burgers' equation, Discontinuous Galerkin methods, Recovery method, pane;ty methods, superconvergence.

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3. *The combination of RDG with upwind DG for advection (URDG).* This is accomplished by using the representation of the solution in terms of the original discontinuous basis when computing inviscid fluxes (with a Riemann solver), while using the representation in terms of the recovery basis to compute the diffusive fluxes. Numerical tests were carried out based on the linear advection-diffusion equation and on Burgers' equation.

4. *Writing the general 1-D form of RDG as a penalty method.* After rewriting the RDG formulas for a number of polynomial orders as a classical method with additional penalty terms with unique coefficient values, the general form of the expansion became evident. The penalty terms are different for odd and even polynomial-space degrees $k$; they contain the product of the jump $[u^{(k)}]$ of the $k^{\text{th}}$ derivative of the solution at the interface with the jump of either the test function $\phi$ (for even $k$) or its derivative $\phi_x$ (for odd $k$).

5. *Dissemination via conferences and publications.* The results of the project were presented at the 2007 AIAA CFD Conference in Miami and the 2007 International Conference on Spectral and Higher-Order Methods (ICOSAHOM 08) in Beijing. The Miami presentation is contained in an AIAA paper [1], the Beijing presentation is to appear in a special issue of *Communications in Computational Physics*[2] in 2008.

6. Technology transfer. By leveraging interactions with WPAFB in the framework of the Michigan/Air Force/Boeing Collaborative Center for Aero Sciences, we were able to get in contact with a representative of HyperComp (Westlake Village, CA), and formulate an STTR announcement. This was adopted and issued by the Computational Mathematcs program of AFOSR. (NB: The ensuing HyperComp/UMich-alliance STTR proposal was indeed succsessful.)

# 3 Accomplishments in the entire grant period

Over the total grant period the RDG method evolved from a promising one-dimensional Discontinuous Galerkin discretization technique for diffusion terms with superior accuracy and good stability to an efficient multidimensional methodology with a solid theoretical underpinning, and ready for transfer to Air Force/industrial use. A comprehensive list list of accomplishments is given below.

*(A) Accuracy in one dimension.* Consider a uniform spatial mesh of open elements $\Omega_j = ]jh, (j+1)h[$. Assume the approximate solution $u$ is represented in each element on a hierarchical polynomial basis of degree $k$. If we want to compute the diffusive fluxes at the interface between $\Omega_e$ and $\Omega_{e+1}$, the recovery principle says we must recover a smooth solution approximation $f$ on $\Omega_e \bigcup \Omega_{e+1}$; this function lies in a polynomial space of degree $2k+1$. The discontinuous solution on $\Omega_e$ and $\Omega_{e+1}$ is the $L_2$ projection of $f$.

The degree of $f$ suggests that the spatial order of accuracy of the resulting DG operator will be $2k+2$; eigenvalue analysis and numerical experiments show that the order is higher.

Specifically, we found that the accuracy of the cell average of the solution is of the order $3k+2$ for $k$ even, and $3k+1$ for $k$ odd.

When solving a 1-D Poisson problem in the piecewise polynomial space with $k \geq 2$, the coefficients of the polynomials of order $\leq\ k-2$ are obtained *exactly*, i. e., within round-off [1].

*(B) Accuracy in two dimensions.* The property of $(k-2)$-exactness is lost in two and more dimensions. We verified numerically, by solving a 2-D Poisson problem on a uniform rectangular grid, that the order of accuracy is $2k+2$. It has been communicated to us by Hung Huynh (NASA Glenn RC) that with the use of a tensor-product basis, rather than the minimal complete basis of order $k$, the 2-D method inherits the order of accuracy of the 1-D method. We have not yet confirmed this.

*(C) Boundary treatment and accuracy.* At the boundary of a 1-D domain there are only $k+2$ data available for recovery, if one just includes the cell on the boundary. The resulting reduction in accuracy of the boundary fluxes contaminates the accuracy in the interior of the domain; this has been confirmed in 1-D numerical experiments The loss can be avoided by allowing the missing $k$ data to come from the next cell inward, starting with the lowest-degree information [1].

This approach also works on multidimensional grids, for recovery along the normal to the domain boundary, as has been confirmed in 2-D numerical experiments on a rectangular grid [1]. At the end of the grant period we started experimenting with the extension of this technique to triangular grids.

*(D) Two-dimensional recovery.* When applying the recovery principle in two dimensions, it is beneficial to switch to a coordinate system aligned with the interface across which the diffusive fluxes must be calculated. In the $\xi$-direction, normal to the interface, the recovery problem resembles 1-D recovery. If in the neighboring elements the solution is represented by a complete 2-D form of degree $k$, the $\xi$-dependence of the recovered smooth solution will be of the degree $2k+1$. In the $\eta$-direction, along the interface, there is no increase in accuracy: the degree remains $k$ [1].

*(E) Stability.* A general stability proof for recovery-based spatial DG operators is still missing, but numerical evaluations of eigenvalues using arbitrarily high-precision arithmetics have confirmed stability up to $k = 6$. In two dimensions stability has been shown on rectangular and right-triangular grids for $k = 1$. If, on a rectangular grid, tensor-product basis functions are chosen (we have not experimented with this), 1-D stability implies 2-D and 3-D stability.

*(F) The recovery basis.* For convenience consider again the 1-D case. When applying the recovery principle to each of the original basis functions on $\Omega_e \bigcup \Omega_{e+1}$ (these are polynomial on one element and zero on the other one), smooth functions result that may be used as basis functions in which to express any recovered smooth solution on $\Omega_e \bigcup \Omega_{e+1}$. To make the *recovery polynomial basis* unique we require:

1. the discontinuous basis functions $\phi_i$ are orthonormal;

2. the recovered basis functions $\psi_j$ are orthonormal with respect to the discontinuous basis functions, in the following sense:

$$\int_{\Omega_e \bigcup \Omega_{e+1}} \phi_i \psi_j dx = \begin{cases} 1, & i = 1, ..., 2k+2,\ j = 1, ..., 2k+2, \quad i = j; \\ 0, & i = 1, ..., 2k=2,\ l = 1, ..., 2K=2, \quad i \neq j. \end{cases}$$

With the above normalizations, the expansion of $f$ in terms of the recovery basis on $\Omega_e \bigcup \Omega_{e+1}$ is *identical* to the expansion of $u$ in terms of the discontinuous basis [3, 2]:

$$u(x) = \sum_{i=1}^{2k+1} a_i \phi_i(x) \implies f(x) = \sum_{i=1}^{2k+1} a_i \ (\psi_i(x),\ x \in \Omega_e \bigcup \Omega_{e+1}. \tag{1}$$

While the discontinuous basis figuring above is hierarchical, with maximum degree $k$, all recovered basis functions are polynomials of degree $2k+1$. Examples of both bases are found in Figures 1 and 2.

When switching from one discontinuous basis to another, the transformation between the bases applies equally to the recovery bases.

*(G) Flux formulas in one dimension.* For the computation of the diffusive fluxes the interface

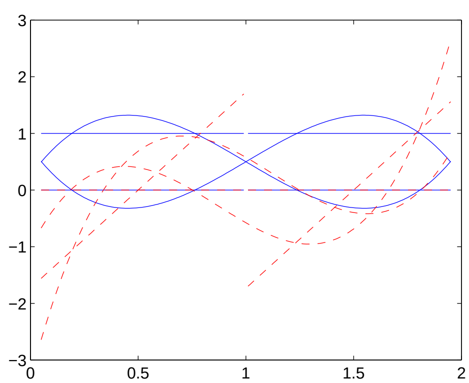


Figure 1: The discontinuous and recovery basis for $k = 1$ and cells $\Omega_0$ and $\Omega_1$ with $h = 1$.

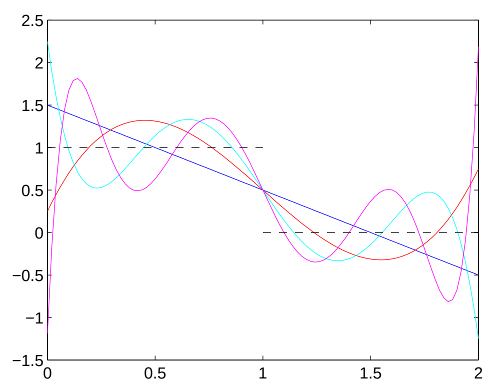


Figure 2: The piecewise constant function and its counterpart in the recovery basis for $k = 0$ to 3.

values of $f$ and $f_x$ are needed. Since $f$ is expressed in the recovery basis functions, and these span the polynomial space of degree $2k+1$ on $\Omega_e \bigcup \Omega_{e+1}$, $f$ is independent of the discontinuous basis used in the elements. It then follows that the diffusive fluxes are also independent of the basis used; only the degree $k$ matters. The fluxes can be uniquely expressed in terms of jumps and averages of the discontinuous solution and its derivatives at the cell interface.

The fundamental theorem of recovery makes it attractive to compute and store the recovery basis for each interface once and for all at the start of a numerical experiment, if on a fixed grid. To compute a diffusion flux at an interface no recovery process is needed; one uses the recovery basis in combination with the coefficients of the discontinuous solution representation on the neighboring elements. This makes RDG computationally efficient. At the end of the grant period we were in the process of implementing RDG in the above way on an unstructured triangular grid.

*(H) Bilinear forms.* The fluxes go directly into bilinear forms. For $k = 1$ the form looks like the symmetric DG method, augmented with Arnold's interior penalty term and a new penalty-like term based on $[\phi_x][u_x]$. For $k = 2$ another penalty term appears, based on $[\phi][u_{xx}]$, Continuing this process for some higher values of $k$, the general form of the expansion became evident. The penalty terms are different for odd and even polynomial-space degrees $k$; they contain the product of the jump $[u^{(k)}]$ of the $k^{\text{th}}$ derivative of the solution at the interface with the jump of either the test function $\phi$ (for even $k$) or its derivative $\phi_x$ (for odd $k$).

The asymmetry of the penalty terms starting with $k = 2$ leads to the appearance of complex eigenvalues with negative real part in the spatial DG operator, rather than purely negative-real eigenvalues. This does not appear to affect the stability of the RDG method.

# 4 Numerical illustrations

The following figures illustrate some of the findings listed in Section 3.

Figure 3, taken from [1], shows the mesh convergence of solution properties for a 1-D Poisson problem, with $k = 2$. The error norms of the solution coefficients $a_3 \equiv \overline{\Delta^2 u}$ (undivided second derivative) and $a_2 \equiv \overline{\Delta u}$ (undivided average gradient) indicate $6^{\text{th}}$- and $7^{\text{th}}$-order accuracy; for $a_1 \equiv \bar{u}$ (cell average) we would expect $8^{\text{th}}$-order accuracy, but the errors are machine-zero for any mesh width.

The property of $(k-2)$-exactness, demonstrated above in one dimension, does not carry over to multiple dimensions. Figure 4 shows properties of the solution to a 2-D Poisson problem on $[0,1] \times [0,1]$ with Dirichlet boundary data. The exact solution is

$$U(x, y) = \frac{1}{2}\{\cos(2\pi x) + \cos(2\pi y) - 1\}. \tag{2}$$

The numerical solution was taken to be piecewise linear ($k = 1$ in both directions) on square meshes, and obtained by marching the associated unsteady problem till convergence. The boundary was accurately treated according to Section 3, item (C). The left plot shows that the method is $4^{\text{th}}$-order accurate, even when measured in the maximum norm. This order is consistent with $2k + 2$.

The solution coefficients and errors shown on the right are for the $16 \times 16$ mesh. It is worth mentioning that the numerical solution exhibits all 4 symmetry axes of the exact solution.

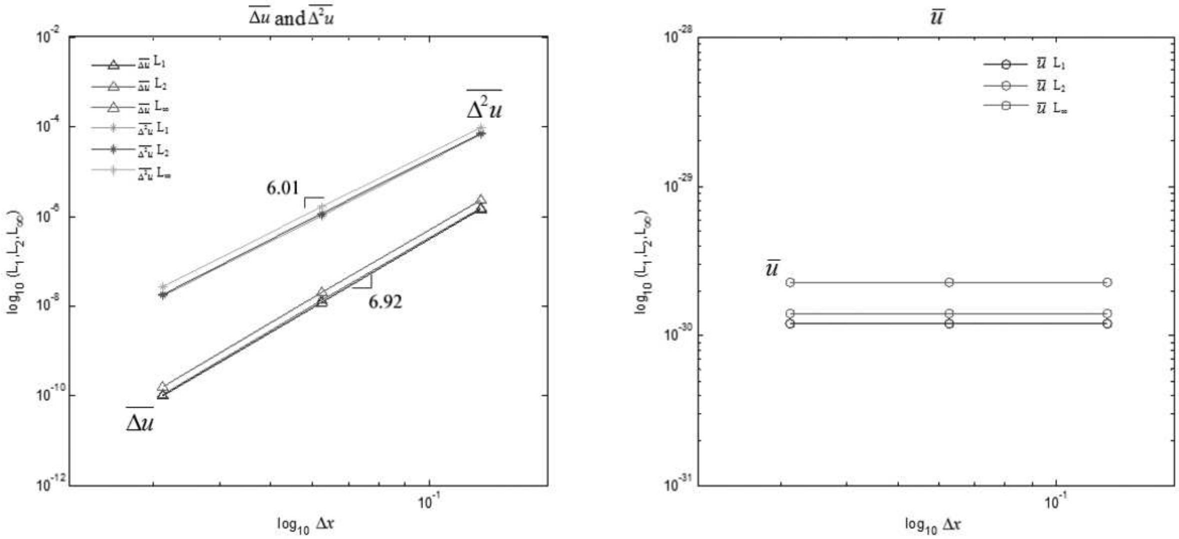


Figure 3: Mesh convergence for 1-D Poisson problem; $k = 2$. Left: convergence with mesh size of error norms of solution coefficients $\overline{\Delta^2 u}$ and $\overline{\Delta u}$. Right: convergence of $\bar{u}$.

Next, our most advanced result on stability. On a structured grid of right triangles, which lends itself to Fourier analysis, we have numerically obtained the DG operator's eigenvalues for $k = 1$; all are non-positive real. Stencil and spectrum are shown in Figure 5.

Finally, results for nonlinear advection-diffusion, *viz.*, Burgers' equation,

$$\partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) = \mu \partial_{xx} u, \tag{3}$$

are just becoming available. We are using the test cases of Wang [4], in order to facilitate comparison with his Spectral Volume method. Figure 6(b) shows the steepening of a compression wave in time for $\mu = 1$ according to the exact solution

$$u(x,t) = \frac{2\sinh(x)}{\cosh(x) + \exp(3-t)}; \tag{4}$$

we computed this solution with $k = 1$ on a sequence of grids and plotted the convergence of the $L_1$ error with mesh width for the final time $t = 4$ (left). The scheme appears to be $4^{\text{th}}$-order accurate, which is surprising since upwind DG for advection is know to be only of order $2k + 1$, that is, 3. Apparently the diffusion error dominates for the meshes chosen, as the diffusion constraint on the time step drives the CFL number to smaller values on finer meshes.

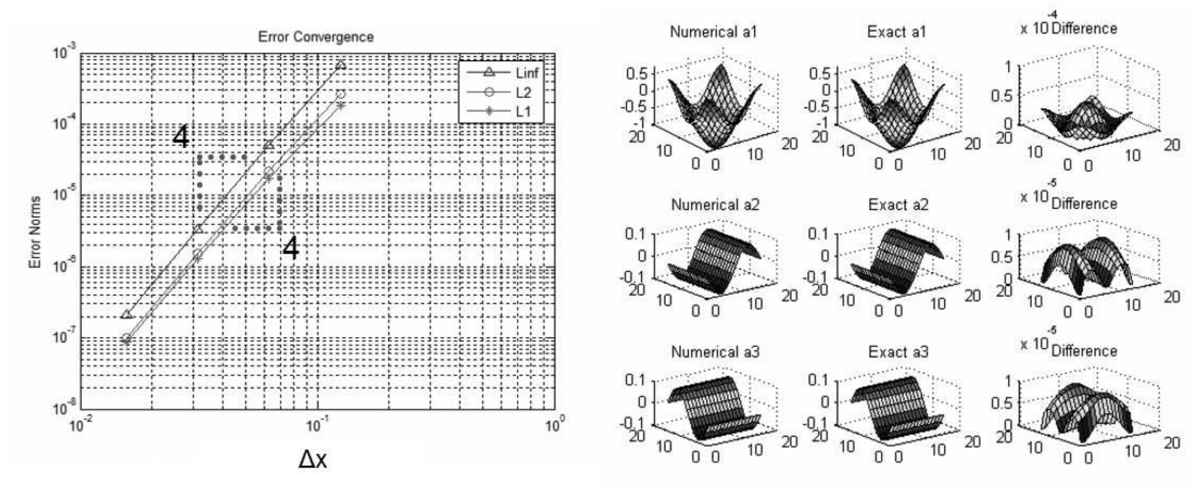


Figure 4: Mesh convergence for 2-D Poisson problem; $p = 1$. Left: convergence with mesh size of error norms of the mesh average $a_1 \equiv \bar{u}$. Right: computed solution coefficients $a_1$ (average value), $a_2$ (average $x$-derivative) and $a_3$ (average $y$-derivative); exact solution coefficients; coefficient errors.

# 5 Acknowledgement/Disclaimer

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# References

[1] B. van Leer, M. Lo, and M. van Raalte, "A Discontinuous Galerkin method for diffusion based on recovery," AIAA Paper 2007-4083, 2007. presented at the 18th AIAA Computational Fluid Dynamics Conference, 25–28 June 2007, Miami, FL.

[2] M. Raalte and B. van Leer, "Bilinear forms for the recovery-based discontinuous Galerkin method for diffusion," *Communications in Computaional Physics*, 2008. to appear.

[3] M. Raalte and B. van Leer, "Bilinear forms for the recovery-based discontinuous galerkin method for diffusion," Tech. Rep. MAS-E0707, Center for Mathematics and Informatics (CWI), 2007.

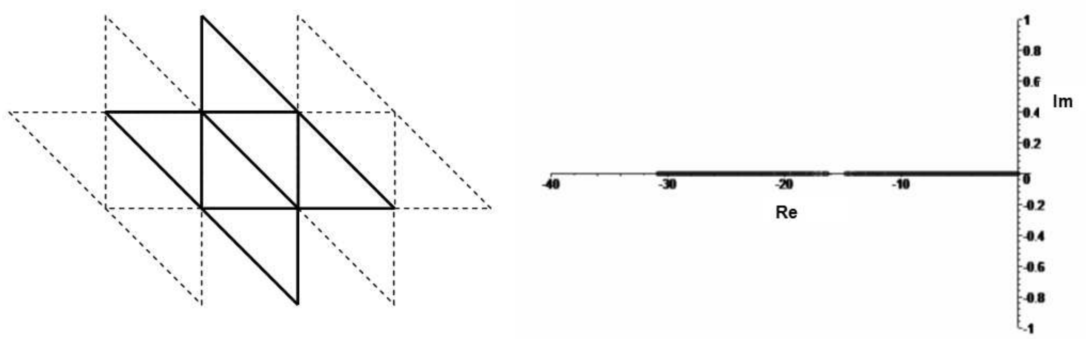



Figure 5: Stability on a triangular grid; $k = 1$ in both directions. Left: stencil of right triangles used in the Fourier analysis of the recovery-based scheme. Right: eigenvalues in the complex plane.

[4] Y. Sun and Z. J. Wang, "High-order spectral volume method for the navier-stokes equations on unstructured grids," AIAA Paper 2004-2133, 2004.

# 6 Personnel supported

Bram van Leer, Professor (PI)
Yoshifumi Suzuki, doctoral student
Marcus Lo, doctoral student
Marc H. van Raalte, postdoctoral researcher, October 2006 - January 2007, visiting from Center for Mathematics and Informatics, Amsterdam, The Netherlands.

# 7 Publications

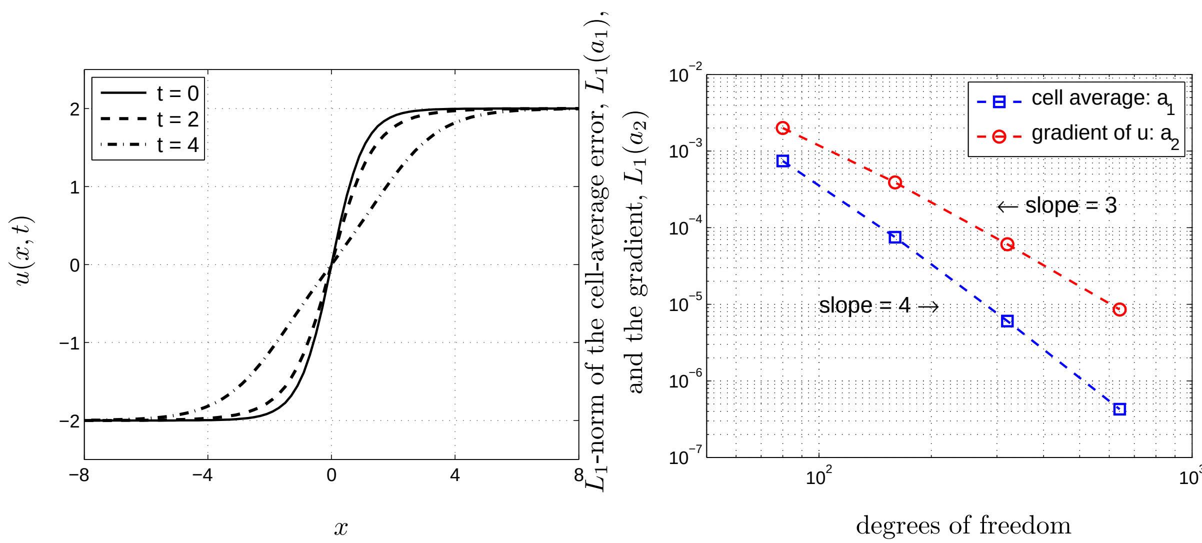
See References 1, 2 and 3.

# 8 Honors and awards

B. van Leer was appointed to Arthur B. Modine Professor of Aerospace Engineering per September 1, 2007. B. van Leer's lifetime achievements inlude an Honorary Doctorate from the Free University of Brussels (1990); he was made a Fellow of the AIAA in 1995.

# 9 AFRL point of contact

Datta Gaitonde, AFRL/WPAFB




(a) Exact solution at different times.

(b) Grid-convergence study at $t = 4$.

Figure 6: Steepening of a compression wave according to Burgers' equation. used as a test case for the DG method with $k = 1$. The time integrator was a 5-stage $4^{\text{th}}$-order Runge-Kutta method.

# 10 Interactions/transitions

(a) *Conference presentations:*

1. NIA Workshop on Spectral and High-Order Methods, Hampton, VA, November 2006 (speaker: B. van Leer);
2. ICFD 2007, Reading, UK, March 2007 (speaker: M. van Raalte);
3. ICOSAHOM 2007, Beijing, China, June 2007 (speaker: M. Lo);
4. 18th AIAA CFD Conference, Miami, FL, June 2007 (speaker: B. van Leer).

(b) *Consulting/Advising:* None.

(c) *Transition:* Development with HyperComp (Westlake Village, CA) of STTR proposal titled "High-order modeling of applied multi-physics phenomena;" awarded in May 2008.

# 11 New discoveries

None patentable.